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ON THE INFLUENCE OF INITIAL STRESSES ON THE VELOCITIES OF THE STONELEY WAVES*

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Wave propagation along a plane boundary separating compressible, previously deformed bodies with elastic potential of arbitrary form, is studied. The linearized theory of wave propagation in bodies with finite initial deformation is used. A case in which one of the bodies is a liquid, is studied. It is shown that in the case of the Murnaghan and harmonic type potentials the wave velocities depend linearly on the initial stresses. In contrast with the case of an unbounded isotropic body /1/, here the character of the dependence is not influenced by the choice of the form of the potential. In the absence of the initial stresses the relations obtained coincide with the results of /2/.

1. Formulation of the problem. We consider a problem of propagation of a surface wave along a plane boundary separating elastic prestressed half-spaces. We assume the half-spaces to be compressible and isotropic, with elastic potentials of arbitrary form. We shall use the Lagrangian (x_1, x_2, x_3) -coordinate system coinciding, in the undeformed state, with the Cartesian coordinate system. We denote the quantities referring to the initial (deformed) state by a zero superscript. We assume that the plane $x_2 = 0$ is the boundary separating the half-spaces. In this case the investigation of the surface waves reduces to the question of solving the plane problem of the linearized theory of elasticity in the Ox_1x_2-plane

$$u_1 = u_1(x_1, x_2, \tau), \quad u_2 = u_2(x_1, x_2, \tau), \quad u_3 \equiv 0 \tag{1.1}$$

According to /3/, when the initial state is homogeneous

$$u_m^\circ = \delta_{im}(\lambda_i - 1)x_i, \quad \lambda_i = \text{const}\ (i, m = 1, 2, 3) \tag{1.2}$$

the above problem of the linearized theory of elasticity reduces to that of solving a boundary value problem for the equation of motion, with boundary conditions in terms of the stresses

$$L_{ij}u_j = 0 \quad (i, j = 1, 2), \quad x_2 = \text{const}, \ \sigma_{22}^*\lambda_2 + \sigma_{22}^{*\circ}\frac{\partial u_2}{\partial x_2} = P_2^*, \ \sigma_{21}^*\lambda_1 + \sigma_{22}^{*\circ}\frac{\partial u_1}{\partial x_2} = P_1^* \tag{1.3}$$

Here L_{ij} denote the differential operators, P_i^* are the components of the perturbations in the external load at $x_2 = \text{const}$, and λ_i are the extension coefficients along the principal axes of the Green's deformation tensor.

If the initial state of stress is determined by the expressions

$$\sigma_{11}^{*\circ} \equiv \sigma_{22}^{*\circ} \neq 0, \quad \sigma_{33}^{*\circ} \neq 0, \quad \lambda_1 \equiv \lambda_2, \quad \lambda_3 \neq 0 \tag{1.4}$$

then the general solution of the system of equations (1.3) can be written in terms of the function χ as follows /3/:

$$u_1 = \left\{\lambda_2^2\left[(\mu_{12} + \sigma_{11}^{*\circ}\lambda_2^{-2})\frac{\partial^2}{\partial x_1^2} + (a_{22} + \sigma_{22}^{*\circ}\lambda_2^{-2})\frac{\partial^2}{\partial x_2^2}\right] - \rho\frac{\partial^2}{\partial \tau^2}\right\}\chi, \quad u_2 = -\lambda_1\lambda_2(\mu_{12} + a_{12})\frac{\partial^2 \chi}{\partial x_1 \partial x_2} \tag{1.5}$$

The expressions for determining the quantities a_{ij} and μ_{12} in terms of the elastic potential and the equation for determining the function χ, are given in /3/.

In addition to the Lagrangian coordinate system we introduce, in the initial state, the system $z_i = \lambda_i x_i$ of coordinates, and will denote all quantities referring to the half-space $z_2 < 0$ by a prime. The displacement in each half-space as well as the normal and tangential surface load intensity perturbations at $z_2 = 0$ are given, respectively, by the formulas (1.5) and (1.3) in which the passage to the variables z_i has been carried out.

The following conditions of continuity must hold at the boundary separating the half-spaces:

$$u_1 = u_1', \ u_2 = u_2', \quad p_1 = p_1', \quad p_2 = p_2' \tag{1.6}$$

Here $p_i = (\lambda_1\lambda_3)^{-1}P_i^*$ and $p_i' = (\lambda_1'\lambda_3')^{-1}P_i^{*\prime}$ are the surface load intensity perturbations at $z_2 = 0$, measured over unit surface area in the initial state.

2. Dispersion relations. We shall obtain dispersion relations describing the

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Stoneley waves at the boundary separating the elastic half-spaces and at the boundary separating a liquid from an elastic half-space. We assume the elastic half-spaces to be previously deformed.

Waves at the boundary separating elastic half-spaces. We consider the case when the initial state of stress satisfies the relations

$$\sigma_{11}^{*\circ} = \sigma_{22}^{*\circ} = \sigma_{11}^{*\circ\prime} = \sigma_{22}^{*\circ\prime} \neq 0, \quad \sigma_{33}^{*\circ} \neq 0, \quad \sigma_{33}^{*\circ\prime} \neq 0 \tag{2.1}$$

We choose the function χ for the half-space $z_2 < 0$ in the form of a surface wave propagating in the positive direction of the Oz_1-axis

$$\chi = \varphi(z_2) \exp[iq(z_1 - C\tau)] \tag{2.2}$$

Here q is the wave number and C is the phase velocity of the wave.

Taking into account (1.5) and (2.2), we obtain from (1.3) the following expression for the unknown function $\varphi(z_2)$:

$$\varphi = Ae^{q\alpha_1 z_2} + Be^{q\alpha_2 z_2}, \quad \alpha_1 = (1 - C^2/C_{l11}^2)^{1/2}, \quad \alpha_2 = (1 - C^2/C_{s12}^2)^{1/2} \tag{2.3}$$

Here A and B are the integration constants, C_{l11} and C_{s12} denote the corresponding velocities of the longitudinal and transverse waves moving in the prestressed body along the Oz_1-axis /1/. Consequently the expressions for the displacements become

$$u_1 = Kq^2 (Ae^{q\alpha_1 z_2} + B\alpha_2^2 e^{q\alpha_2 z_2}) \exp[iq(z_1 - C\tau)], \quad u_2 = -iKq^2 (A\alpha_1 e^{q\alpha_1 z_2} + B\alpha_2 e^{q\alpha_2 z_2}) \exp[iq(z_1 - C\tau)] \tag{2.4}$$

$$K = \rho(C_{l11}^2 - C_{s12}^2)$$

For the half-space $z_2 > 0$ the function χ' and the displacements u_i' $(i = 1, 2)$ are given by the formulas obtained from (2.2)–(2.4) where $A, B, \alpha_1, \alpha_2, C_{l11}, C_{s12}, \rho, K$ has been replaced by $A', B', \alpha_1', \ldots, K'$ and $\exp(q\alpha_i z_2)$ by $\exp(-q\alpha_i' z_2)$. In addition, a plus sign must be used in the expression for u_2'. Taking into account, for the half-space $z_2 < 0$, the expressions for the displacements (2.4) as well as the equations of state /3/

$$\sigma_{21}^* = \lambda_1\lambda_2\mu_{12}\left(\frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1}\right), \quad \sigma_{22}^* = a_{12}\lambda_1^2 \frac{\partial u_1}{\partial z_1} + a_{22}\lambda_2^2 \frac{\partial u_2}{\partial z_2} \tag{2.5}$$

we obtain, for the surface load intensity perturbations at the boundary $z_2 = 0$, the following expressions in accordance with (1.3):

$$p_1 = Kq^3 (\lambda_1^2\lambda_3)^{-1} [(\rho C_{s12}^2 + \mu_{12}\lambda_1^4)\alpha_1 A + (\rho C_{s12}^2 \alpha_2^2 + \mu_{12}\lambda_1^4)\alpha_2 B] \tag{2.6}$$

$$p_2 = iKq^3 (\lambda_1^2\lambda_3)^{-1} [(a_{12}\lambda_1^4 - \rho C_{l11}^2 \alpha_1^2) A + (a_{12}\lambda_1^4 - \rho C_{l11}^2)\alpha_2^2 B]$$

where the multiplying factor $\exp[iq(z_1 - C\tau)]$ has been omitted.

For the half-space $z_2 > 0$ the quantities p_i' $(i = 1, 2)$ can be obtained from the formulas identical to (2.6) in which $\lambda_1, \lambda_3, a_{12}, \mu_{12}, C_{s12}, C_{l11}, K, \rho$ have been replaced by $\lambda_1', \lambda_3', a_{12}', \ldots, \rho'$. Here the minus sign must be used in the expression for p_1'.

Using the boundary conditions (1.6) and taking into account the expressions (2.4) and (2.6), we arrive at the following characteristic equation describing the propagation of the Stoneley waves through a prestressed medium:

$$C^4 [(\rho - \rho' m)^2 - (\rho\alpha_1' + \rho'\alpha_1 m)(\rho\alpha_2' + \rho'\alpha_2 m)] + 2SC^2 (\rho\alpha_1'\alpha_2' - \rho'\alpha_1\alpha_2 m - \rho + \rho' m) + S^2 (\alpha_1\alpha_2 - 1)(\alpha_1'\alpha_2' - 1) = 0, \quad m = \lambda_1^2\lambda_3 (\lambda_1'^2\lambda_3')^{-1}, \quad S = \rho C_{s12}^2 + \mu_{12}\lambda_1^4 - m(\rho' C_{s12}'^2 + \mu_{12}'\lambda_1'^4) \tag{2.7}$$

In the absence of the initial stresses the characteristic equation (2.7) assumes its classical form /4/.

Waves at the boundary between a liquid and an elastic half-space. Let us consider the case when one of the half-spaces, e.g. $z_2 < 0$, is a liquid $(C_{s12} = 0, C_{l11} = c_1, \alpha_2 = \infty$ and c_1 is the speed of sound in the liquid), and the initial state of stress is described by

$$\sigma_{11}^{*\circ} = \sigma_{22}^{*\circ} = \sigma_{33}^{*\circ} = 0, \ \sigma_{11}^{*\circ\prime} = \sigma_{22}^{*\circ\prime} \neq 0, \ \sigma_{33}^{*\circ\prime} \neq 0 \tag{2.8}$$

In this case the characteristic equation (2.7) becomes

$$(S + m\rho' C^2)^2 \alpha_1 + m\rho\rho'\alpha_1' C^4 - S^2\alpha_1\alpha_1'\alpha_2' = 0, \quad m = (\lambda_1'^2\lambda_3')^{-1}, \quad S = -m(2\mu_{12}'\lambda_1'^4 + \sigma_{11}^{*\circ\prime}\lambda_1'^2) \tag{2.9}$$

which describes the propagation of the Stoneley waves along the boundary between the fluid and prestressed half-space. The classical case /4/ is obtained by putting in (2.9) $\sigma_{11}^{*\circ\prime} = \sigma_{22}^{*\circ\prime} = \sigma_{33}^{*\circ\prime} = 0$.

3. **Examples**. We shall consider some examples for the bodies with particular forms of the elastic potentials.

Waves at the boundary separating elastic half-spaces. Consider the case when the initial state of stress is described by the expressions

$$\sigma_{11}^{*\circ} = \sigma_{22}^{*\circ} = \sigma_{33}^{*\circ} = \sigma_{11}^{*\circ\prime} = \sigma_{22}^{*\circ\prime} = 0, \quad \sigma_{33}^{\circ\prime} = p^\circ, \quad \lambda_1 = \lambda_2 = \lambda_3 = 1 \tag{3.1}$$

In this case we write the equation (2.7) in the form

$$v^2 [(\rho / \rho' - m)^2 - (\beta_1 \rho/\rho' + m\beta_2)(\beta_3 \rho/\rho' + m\beta_4)] + 4v\,(s\rho/\rho' - mg)\times$$
$$(\beta_1\beta_3\rho/\rho' - m\beta_2\beta_4 - \rho/\rho' + m) + 4\,(s\rho/\rho' - mg)^2 \times (\beta_2\beta_4 - 1)(\beta_1\beta_3 - 1) = 0 \tag{3.2}$$

$$\beta_1 = (1 - v/d)^{1/2},\ \beta_2 = (1 - v/r)^{1/2},\ \beta_3 = (1 - v/g)^{1/2},\ \ \beta_4 = (1 - v/s)^{1/2},\ v = (C/c_2')^2,\ g = (C'_{s12}/c_2')^2$$
$$d = (C'_{l11}/c_2')^2,\ r = (c_1/c_2')^2,\ s = (c_2/c_2')^2,\ c_1^2 = (\lambda + 2\mu)/\rho,\ c_2^2 = \mu/\rho,\ c_2'^2 = \mu'/\rho',\ m = (\lambda_1'^2\lambda_3')^{-1} \tag{3.3}$$

where λ, λ' and μ, μ' are the Lamé constants.

1°. Using the framework of the theory of finite initial deformations, we consider the case for a half-space with harmonic-type potential /5/

$$\Phi^\circ = {}^1/_2\lambda S_1^{\circ 2} + \mu S_2^\circ,\ \ S_1^\circ = (\lambda_1 - 1) + (\lambda_2 - 1) + (\lambda_3 - 1),\ \ S_2^\circ = (\lambda_1 - 1)^2 + (\lambda_2 - 1)^2 + (\lambda_3 - 1)^2 \tag{3.4}$$

Taking into account /6/ we obtain, from (3.3) and (3.4),

$$n = 0,\ \lambda_1' = 1,\ \ n \neq 0,\ \ \lambda_1' = \{[(3t - 4) + n\,(t - 2)(6t - 8)]^{1/2} - 3t + 4\}/[n\,(t - 2)] \tag{3.5}$$

$$g = \lambda_1' - {}^1/_2\,(t - 2)\,\lambda_1'\,[2\,(\lambda_1' - 1) + (\lambda_3' - 1)],\ n = p^\circ/\mu',\ \ d = \lambda_1'\,\{2 + (t - 2)\,[\lambda_1' - 2\,(\lambda_1' - 1) - (\lambda_3' - 1)]\},\ t = (c_1'/c_2')^2$$

Solving (3.2) numerically we find, that the dependence of $\eta = (C - C_0)/C_0$ on n has linear character $\eta = kn$ (C_0 is the velocity of the Stoneley wave in a stress-free body). For the materials combination of the alloy AMG-6 ($\rho = 2.63 \cdot 10^3$ kg/m^3, $\lambda = 49.6 \cdot 10^9$ Pa, $\mu = 24.8 \cdot 10^9$ Pa) is steel 45G17IU3 ($\rho' = 7.54 \cdot 10^3$ kg/m^3, $\lambda' = 78.0 \cdot 10^9$ Pa, $\mu' = 63.8 \cdot 10^9$ Pa) $k = -0.088$ ($-0{,}5 \cdot 10^{-2} \leqslant n \leqslant 0.5 \cdot 10^{-2}$) and for the combination of the alloy AMG-6 is steel 09G2S ($\rho' = 7.795 \cdot 10^3$ kg/m^3, $\lambda' = 90.75 \cdot 10^9$ Pa, $\mu' = 75.95 \cdot 10^9$ Pa) $k = -0.0714$ ($-3.5 \cdot 10^{-3} \leqslant n \leqslant 3.5 \cdot 10^{-3}$).

2°. Consider the case when the materials occupying the half-spaces are described by the Murnaghan potential /7/

$$\Phi^\circ = {}^1/_2\lambda A_1^{\circ 2} + \mu A_2^\circ + {}^1/_3 a A_1^{\circ 3} - b A_1^\circ A_2^\circ + {}^1/_3 c A_3^\circ \tag{3.6}$$

Here a, b and c are the third order elastic constants and A_i ($i = 1, 2, 3$) are the algebraic invariants of the Green's deformation tensor. We shall limit ourselves to the linear approximation /1/. In this case we assume that $\sigma_{33}^{*\circ\prime} = \sigma_{33}^{\circ\prime} = p^\circ$ and determine the quantities (3.3) using the formulas

$$g = \lambda_1'^{4} + (\mu')^{-1}(b' + {}^1/_2 c')(\lambda_1'^2 - 1)\,\lambda_1'^4 + {}^1/_2\,(\mu')^{-1} b'\,(\lambda_3'^2 - 1)\,\lambda_1'^4$$
$$d = (\lambda' + 2\mu')(\mu')^{-1}\lambda_1'^4 + [(2a' + 4b' + c')(\lambda_1'^2 - 1) + (a' + b')\times$$
$$(\lambda_3'^2 - 1)]\,\lambda_1'^4(\mu')^{-1},\ K_0 = 3\lambda' + 2\mu',\ \lambda_1'^2 = 1 - K_0^{-1}\lambda' n,\ \lambda_3'^2 = 1 + 2K_0^{-1}(\lambda' + \mu')\,n \tag{3.7}$$

The results of the computations for the same materials combinations as used in example 1° show that η depends linearly on n. For the combination of the alloy AMG-6 ($a = 30.2 \cdot 10^{10}$ Pa, $b = -4.8 \cdot 10^{10}$ Pa, $c = -28.6 \cdot 10^{10}$ Pa) is steel 45G17IU3 ($a' = 60.1 \cdot 10^{10}$ Pa, $b' = -21.1 \cdot 10^{10}$ Pa, $c' = -33.5 \cdot 10^{10}$ Pa) $k = -0.185$ and for the combination alloy AMG-6 is steel 09G2S ($a' = -25.5 \cdot 10^{10}$ Pa, $b' = -20.6 \cdot 10^{10}$ Pa, $c' = -46.45 \cdot 10^{10}$ Pa) $k = -0.01$.

Waves at the boundary between a liquid and an elastic half-space. We consider the case when the initial state of stress is described by the expressions (3.1). In this case the characteristic equation (2.9) assumes the form

$$(2 - v/g)^2\beta_2 - \rho v^2\beta_1/(\rho' m g^2) - 4\beta_1\beta_2\beta_3 = 0 \tag{3.8}$$

3°. Using the framework of the theory of finite initial deformations, we consider a problem for an elastic half-space with a harmonic-type potential. The quantities appearing in (3.8) are given by the relations (3.5). Equation (3.8) was solved numerically using the following parameters /8/: $t = 2.857$, $r = 1.176$, $\rho/\rho' = 0.5$. The analysis of the solution shows the linear dependence of η on n $k = -0.12$ ($-0.002 \leqslant n \leqslant 0.002$).

The results obtained make it possible to establish the character of the influence of the initial stresses on the Stoneley waves propagating along the boundary separating two elastic bodies or a liquid and an elastic body. The influence consists of the fact that the velocity of the Stoneley wave increases (decreases) if one of the elastic spaces is comparessed (stretched) in the direction perpendicular to the direction of the wave propagation. The wave velocity depends on the initial stresses in a linear manner.

Analysing the computational results we find, that in contrast to the case of waves in an unbounded isotropic body, the choice of the form of the elastic potential does not influence the character of the dependence of the velocity of the Stoneley waves on the initial stresses. In both cases the velocity increases under compression and decreases under tension.

In conclusion we note that certain results concerning waves at the boundary separating prestressed bodies are given in /9,10/.

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